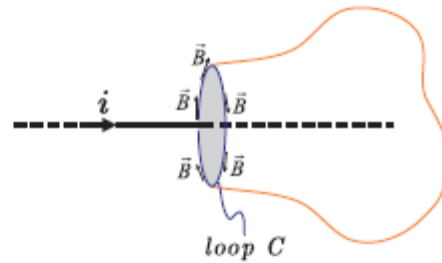


Displacement Current and Maxwell's Equations


Displacement Current


We saw in Chap.7 that we can use **Ampère's law** to calculate magnetic fields due to currents.

We know that the integral $\oint_C \vec{B} \cdot d\vec{s}$ around any close loop C is equal to $\mu_0 i_{incl}$, where i_{incl} = *current passing an area bounded by the closed curve C .*



e.g.


 = Flat surface bounded by loop C


 = Curved surface bounded by loop C

If **Ampère's law** is true all the time, then the i_{incl} *determined should be independent of the surface chosen.*

Let's consider a simple case: *charging a capacitor*.

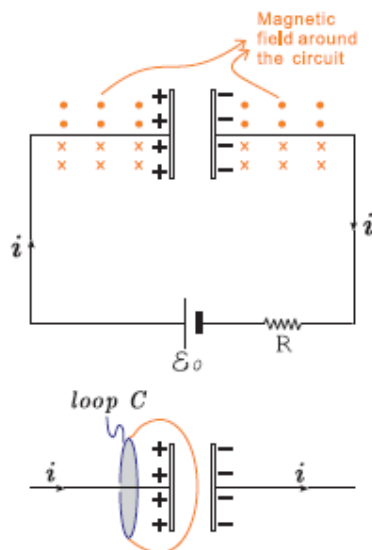
From Chap.5, we know there is a current flowing $i(t) = \frac{\mathcal{E}_0}{R} e^{-t/RC}$, which leads to a magnetic field observed \vec{B} . With Ampère's law, $\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i_{incl}$. BUT WHAT IS i_{incl} ?

If we look at , $i_{incl} = i(t)$

If we look at , $i_{incl} = 0$

(\because There is no charge flow between the capacitor plates.)

\therefore Ampère's law is either WRONG or INCOMPLETE.



Two observations:

1. While there is no current between the capacitor's plates, there is a *time-varying electric field between the plates of the capacitor*.
2. We know *Ampère's law is mostly correct from measurements of B-field around circuits*.

\Downarrow

Can we revise Ampère's law to fix it?

Electric field between capacitor's plates: $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$, where Q = *charge on capacitor's plates*, A = *Area of capacitor's plates*.

$$\therefore Q = \epsilon_0 \underbrace{E \cdot A}_{\text{Electric flux}} = \epsilon_0 \Phi_E$$

\therefore We can define

$$\frac{dQ}{dt} = \epsilon_0 \frac{d\Phi_E}{dt} = i_{disp}$$

where i_{disp} is called **Displacement Current** (first proposed by Maxwell).
Maxwell first proposed that this is the missing term for the Ampère's law:

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \left(i_{incl} + \epsilon_0 \frac{d\Phi_E}{dt} \right) \quad \text{Ampère-Maxwell law}$$

Where i_{incl} = current through any surface bounded by C ,

Φ_E = electric flux through that *same surface bounded by curve C* , $\Phi_E = \int_S \vec{E} \cdot d\vec{a}$.

11.2 Induced Magnetic Field

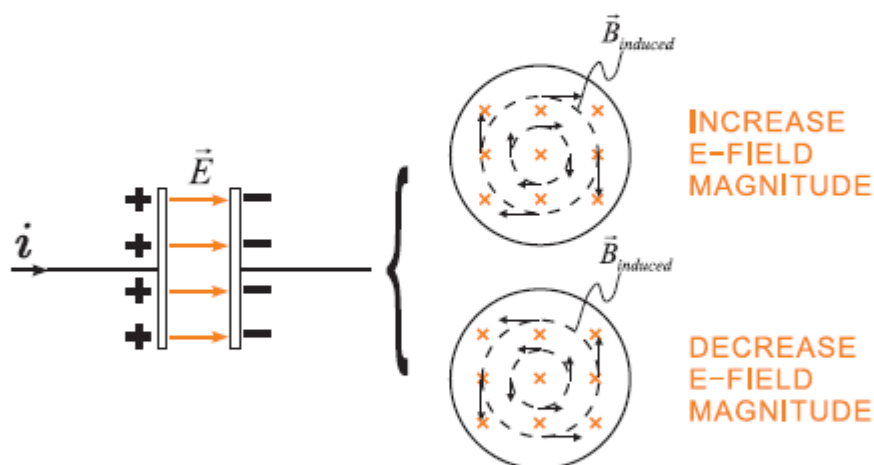
We learn earlier that electric field can be generated by

- $\left\{ \begin{array}{l} \text{charges} \\ \text{changing magnetic flux} \end{array} \right.$

We see from Ampère-Maxwell law that a magnetic field can be generated by

- $\left\{ \begin{array}{l} \text{moving charges (current)} \\ \text{changing electric flux} \end{array} \right.$

That is, a change in electric flux through a surface bounded by C can lead to an *induced magnetic field along the loop C* .



Notes The induced magnetic field is along the *same direction* as caused by the *changing electric flux*.

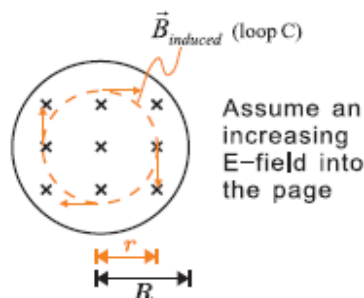
Example What is the magnetic field strength inside a circular plate capacitor of radius R with a current $I(t)$ charging it?

Answer Electric field of capacitor

$$E = \frac{Q}{\epsilon_0 A} = \frac{Q}{\epsilon_0 \pi R^2}$$

Electric flux inside capacitor through a loop C of radius r :

$$\Phi_E = E \cdot \pi r^2 = \frac{Qr^2}{\epsilon_0 R^2}$$



Ampère-Maxwell Law inside capacitor:

$$\underbrace{\oint_C \vec{B} \cdot d\vec{s}}_{\therefore \vec{B}_{induced} \parallel d\vec{s}} = \mu_0(i_{incl} + \epsilon_0 \frac{d\Phi_E}{dt})$$

$$\begin{aligned} \underbrace{2\pi r}_{\text{Length of loop } C} B_{induced} &= \mu_0 \epsilon_0 \frac{d}{dt} \left(\frac{Qr^2}{\epsilon_0 R^2} \right) \\ &= \mu_0 \frac{r^2}{R^2} \underbrace{\frac{dQ}{dt}}_{I(t)} \end{aligned}$$

$$\therefore B_{induced} = \frac{\mu_0 r}{2\pi R^2} I(t) \quad \text{for } r < R$$

Outside the capacitor plate:

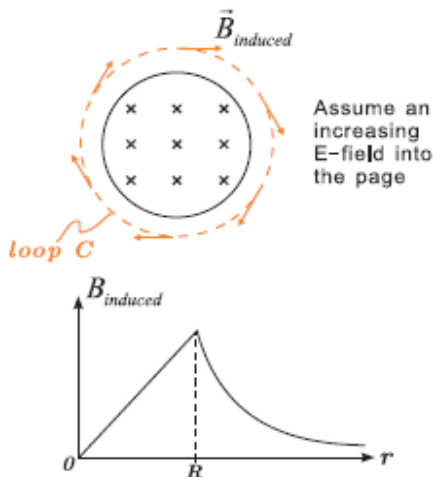
Electric flux through loop C : $\Phi_E = E \cdot$

$$\pi R^2 = \frac{Q}{\epsilon_0}$$

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0(i_{incl} + \epsilon_0 \frac{d\Phi_E}{dt})$$

$$2\pi r B_{induced} = \mu_0 \epsilon_0 \left(\frac{1}{\epsilon_0} \cdot \frac{dQ}{dt} \right)$$

$$\therefore B_{induced} = \frac{\mu_0 I(t)}{2\pi r}$$



Maxwell's Equations

The four equations that *completely* describe the behaviors of electric and magnetic fields.

$$\begin{aligned}
 \oint_S \vec{E} \cdot d\vec{a} &= \frac{Q_{incl}}{\epsilon_0} \\
 \oint_S \vec{B} \cdot d\vec{a} &= 0 \\
 \oint_C \vec{E} \cdot d\vec{s} &= -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a} \\
 \oint_C \vec{B} \cdot d\vec{s} &= \mu_0 i_{incl} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}
 \end{aligned}$$

The one equation that describes *how matter reacts to electric and magnetic fields*.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Features of Maxwell's equations:

- (1) There is a high level of *symmetry* in the equations. That's why the study of electricity and magnetism is also called **electromagnetism**.
There are *small asymmetries* though:

- i) There is *NO point "charge" of magnetism / NO magnetic monopole*.
- ii) Direction of induced E-field *opposes to* B-flux change.
Direction of induced B-field *enhances* E-flux change.

- (2) Maxwell's equations predicted the existence of propagating waves of E-field and B-field, known as **electromagnetic waves (EM waves)**.

Examples of EM waves: visible light, radio, TV signals, mobile phone signals, X-rays, UV, Infrared, gamma-ray, microwaves...

- (3) Maxwell's equations are *entirely consistent with the special theory of relativity*. This is *not* true for Newton's laws!

Electromagnetic theory is a discipline concerned with the study of charges at rest and in motion. Electromagnetic principles are fundamental to the study of electrical engineering and physics. Electromagnetic theory is also indispensable to the understanding, analysis and design of various electrical, electromechanical and electronic systems. Some of the branches of study where electromagnetic principles find application are:

RF communication

Microwave Engineering

Antennas

Electrical Machines

Satellite Communication

Atomic and nuclear research

Radar Technology

Remote sensing

EMI EMC

Quantum Electronics

VLSI

Electromagnetic theory is a prerequisite for a wide spectrum of studies in the field of Electrical Sciences and Physics. Electromagnetic theory can be thought of as generalization of circuit theory. There are certain situations that can be handled exclusively in terms of field theory. In electromagnetic theory, the quantities involved can be categorized as **source quantities** and **field quantities**. Source of electromagnetic field is electric charges: either at rest or in motion. However an electromagnetic field may cause a redistribution of charges that in turn change the field and hence the separation of cause and effect is not always visible.

Electric charge is a fundamental property of matter. Charge exist only in positive or negative integral multiple of **electronic charge**, $-e$, $e = 1.60 \times 10^{-19}$ coulombs. [It may be noted here that in 1962, Murray Gell-Mann hypothesized **Quarks** as the basic building blocks of matters. Quarks were predicted to carry a fraction of electronic charge and the existence of Quarks have been experimentally verified.] Principle of conservation of charge states that the total charge (algebraic sum of positive and negative charges) of an isolated system remains unchanged, though the charges may redistribute under the influence of electric field. Kirchhoff's Current Law (**KCL**) is an

assertion of the conservative property of charges under the implicit assumption that there is no accumulation of charge at the junction.

Electromagnetic theory deals directly with the electric and magnetic field vectors where as circuit theory deals with the voltages and currents. Voltages and currents are integrated effects of electric and magnetic fields respectively. Electromagnetic field problems involve three space variables along with the time variable and hence the solution tends to become correspondingly complex. Vector analysis is a mathematical tool with which electromagnetic concepts are more conveniently expressed and best comprehended. Since use of vector analysis in the study of electromagnetic field theory results in real economy of time and thought, we first introduce the concept of vector analysis.

In this course we consider light to be electromagnetic waves of frequencies ν in the visible range, so that $\nu \simeq (4 - 7.5) \times 10^{14}$ Hz. Since $\lambda = \frac{c}{\nu}$, where c is the speed of light in vacuum ($c \simeq 3 \times 10^8$ m/s), we find that the corresponding wavelength interval is $\lambda \simeq (0.4 - 0.75) \mu\text{m}$. Thus, to study the propagation of light we must consider the propagation of the electromagnetic field, which is represented by the two vectors \mathbf{E} and \mathbf{B} , where \mathbf{E} is the electric field strength and \mathbf{B} is the magnetic induction or the magnetic flux density. To enable us to describe the interaction of the electromagnetic field with material objects we need three additional vector quantities, namely the current density \mathbf{J} , the displacement \mathbf{D} , and the magnetic field strength \mathbf{H} .

Maxwell's equations

The five vectors mentioned above are linked together by Maxwell's equations, which in Gaussian units are

$$\nabla \times \mathbf{H} = \frac{1}{c} \dot{\mathbf{D}} + \frac{4\pi}{c} \mathbf{J}, \quad (1.1.1)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \dot{\mathbf{B}}. \quad (1.1.2)$$

In addition we have the two scalar equations

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \quad (1.1.3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.1.4)$$

where ρ is the charge density. Equation (1.1.3) can be said to *define* the charge density ρ . Similarly, we can say that (1.1.4) implies that free magnetic charges do not exist.

The continuity equation

The charge density ρ and the current density \mathbf{J} are not *independent* quantities. By taking the divergence of (1.1.1) and using that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for an arbitrary vector \mathbf{A} , we find that

$$\nabla \cdot \mathbf{J} + \frac{1}{4\pi} \nabla \cdot \dot{\mathbf{D}} = 0,$$

which on using (1.1.3) gives

$$\nabla \cdot \mathbf{J} + \dot{\rho} = 0. \quad (1.2.1)$$

This equation is called the continuity equation, and it expresses *conservation of charge*. By integrating (1.2.1) over a closed volume V with surface S , we find

$$\iiint_V \nabla \cdot \mathbf{J} dv = - \iiint_V \frac{\partial \rho}{\partial t} dv, \quad (1.2.2)$$

which by use of the divergence theorem gives

$$\oint_S \mathbf{J} \cdot \hat{\mathbf{n}} da = - \frac{d}{dt} \iiint_V \rho dv = - \frac{d}{dt} Q. \quad (1.2.3)$$

Here $\hat{\mathbf{n}}$ is the unit surface normal in the direction out of the volume V , so that (1.2.3) shows that the integrated current flux out of the closed volume V is equal to the loss of charge in the same volume.

Digression 1: Notation

- **Bold face** is used to denote vector quantities, e.g.

$$\mathbf{E} = E_x \hat{\mathbf{e}}_x + E_y \hat{\mathbf{e}}_y + E_z \hat{\mathbf{e}}_z,$$

where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are unit vectors along the axes in a Cartesian co-ordinate system.

- A dot above a symbol is used to denote the time derivative, e.g.

$$\dot{\mathbf{B}} = \frac{\partial}{\partial t} \mathbf{B}.$$

- \mathbf{E} , \mathbf{B} , \mathbf{D} , \mathbf{H} , ρ , and \mathbf{J} are functions of the position \mathbf{r} and the time t , e.g.

$$\mathbf{D} = \mathbf{D}(\mathbf{r}, t).$$

- The connection between Gaussian and other systems of units, e.g. MKS units, follows from J.D. Jackson, "Classical Electrodynamics", Wiley (1962), pp. 611-621. For conversion between Gaussian units and MKS units, we refer to the table on p. 621 in this book.

The material equations

Maxwell's equations (1.1.1)-(1.1.4), which connect the fundamental quantities \mathbf{E} , \mathbf{H} , \mathbf{B} , \mathbf{D} , and \mathbf{J} , are not sufficient to *uniquely* determine the field vectors (\mathbf{E} , \mathbf{B}) from a given distribution of currents and charges. In addition we need the so-called material equations, which describe how the field is influenced by matter.

In general the material equations can be relatively complicated. But if the field is *time harmonic* and the matter is *isotropic* and *at rest*, the material equations have the following simple form

$$\mathbf{J}_e = \sigma \mathbf{E}, \quad (1.3.1)$$

$$\mathbf{D} = \epsilon \mathbf{E}, \quad (1.3.2)$$

$$\mathbf{B} = \mu \mathbf{H}, \quad (1.3.3)$$

where σ is the conductivity, ε is the permittivity or dielectric constant, and μ is the permeability.

Equation (1.3.1) is Ohm's law, and \mathbf{J}_c is the conduction current density, which arises because the material has a non-vanishing conductivity ($\sigma \neq 0$). The total current density \mathbf{J} in (1.1.1) can in addition consist of an externally applied current density \mathbf{J}_0 , so that

$$\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_c = \mathbf{J}_0 + \sigma \mathbf{E}. \quad (1.3.4)$$

Digression 2: General material considerations

- A material that has a non-negligible conductivity σ is called a conductor, while a material that has a negligible conductivity is called an insulator or a *dielectric*.
- Metals are good conductors.
- Glass is a dielectric; $\varepsilon \simeq 2.25$; $\sigma = 0$; $\mu = 1$.
- In anisotropic media (e.g. crystals) the relation in (1.3.2) is to be replaced by $\mathbf{D} = \underline{\underline{\varepsilon}} \mathbf{E}$, where $\underline{\underline{\varepsilon}}$ is a tensor, dyadic or matrix.
- In a plasma (1.3.1) is to be replaced by $\mathbf{J} = \underline{\underline{\sigma}} \mathbf{E}$, where the conductivity is a tensor.
- There are also magnetically anisotropic media, in which (1.3.3) is to be replaced by $\mathbf{B} = \underline{\underline{\mu}} \mathbf{H}$. Thus, in this case the permeability is a tensor. Such materials are not important in optics.
- In dispersive media ε is frequency dependent, i.e. $\varepsilon = \varepsilon(\omega)$. Maxwell's equations and the material equations are still valid for each frequency component or time harmonic component of the field. For a pulse consisting of many frequency components, one must apply Fourier analysis to solve Maxwell's equations and the material equations separately for each time harmonic component, and then perform an inverse Fourier transformation.
- In *non-linear* media there is no linear relation between \mathbf{D} and \mathbf{E} (equation (1.3.2) is not valid). Most media become non-linear when the electric field strength becomes sufficiently high.

1.4 Boundary conditions

Hitherto we have assumed that ε and μ are continuous functions of the position. But in optics we often have systems consisting of several different types of glass. At the transition between air and glass or between two different types of glass the material parameters are discontinuous. Let us therefore consider what happens to the electromagnetic field at the boundary between two media.

Consider two media that are separated by an interface, as illustrated in Fig. 1.1. From Maxwell's equations, combined with Stokes' and Gauss' theorems, one can derive the following boundary conditions

$$\hat{\mathbf{n}} \cdot (\mathbf{B}^{(2)} - \mathbf{B}^{(1)}) = 0, \quad (1.4.1)$$

$$\hat{\mathbf{n}} \cdot (\mathbf{D}^{(2)} - \mathbf{D}^{(1)}) = 4\pi \rho_s, \quad (1.4.2)$$

$$\hat{\mathbf{n}} \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0, \quad (1.4.3)$$

$$\hat{\mathbf{n}} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(1)}) = \frac{4\pi}{c} \mathbf{J}_s, \quad (1.4.4)$$

where $\hat{\mathbf{n}}$ is a unit vector along the surface normal. According to (1.4.1) the normal component of \mathbf{B} is continuous across the boundary, while (1.4.2) says that if there exists a surface charge density

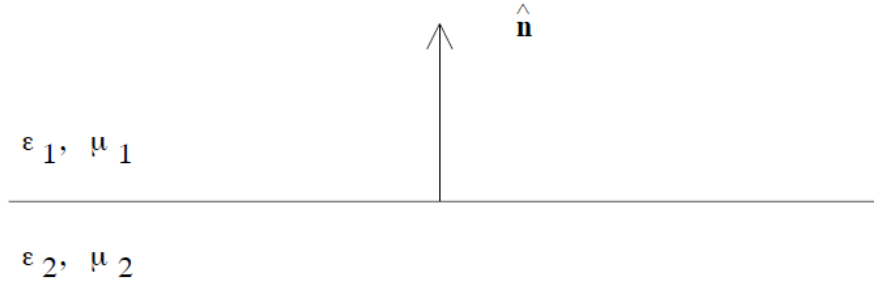


Figure 1.1: A plane interface with unit normal \hat{n} separates two different dielectric media.

ρ_s at the boundary, then the normal component of \mathbf{D} is changed by $4\pi\rho_s$ across the boundary between the two media. According to (1.4.3) the tangential component of \mathbf{E} is continuous across the boundary, while (1.4.4) implies that if there exists a surface current density \mathbf{J}_s at the boundary, then the tangential component of \mathbf{H} , i.e. of $\hat{n} \times \mathbf{H}$, is changed by $\frac{4\pi}{c}\mathbf{J}_s$.

Free-space wave equation

We consider first propagation in a homogeneous, isotropic, nonconducting ($\sigma=0$), source-free ($\rho=0, \mathbf{J}=0$), dielectric medium. ϵ and μ are constants at all points and in all directions in space).

We have:

$$\nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \Rightarrow \nabla \times \vec{B} = \epsilon\mu \frac{\partial \vec{E}}{\partial t}$$

Since a time-varying \mathbf{B} -field gives rise to an \mathbf{E} -field, and vice versa, it may be possible to derive a single differential equation for \vec{E} . We do this by taking:

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \vec{B}) = -\epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{Now, } \nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

But, $\nabla \cdot \vec{E} = 0$, so we have:

$$\nabla^2 \vec{E} = \epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

This is the Maxwell wave equation for the electric field. If one wants to eliminate \vec{E} in Maxwell's equation, one must find the same wave equation for \vec{H} :

$$\nabla^2 \vec{H} = \epsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2}$$

One can solve either wave equation for \vec{E} and \vec{H} , and then use Maxwell's equations to determine the other. It turns out that although virtually all optics publications start with the wave equation for \vec{E} , there are situations (e.g. photonics crystal mode calculations) when it is preferable to solve for \vec{H} first.

Of course, you might ask, what does $\nabla^2 \vec{E}$ even mean? The Laplace operator ∇^2 operates on scalar wave equations, one for each vector component:

$$\nabla^2 \vec{E}_j = \epsilon\mu \frac{\partial^2 \vec{E}_j}{\partial t^2} \quad j=x,y,z$$

Wave equations have the general form

$$\nabla^2 f(\vec{r}, t) = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Where f = scalar wave amplitude

v = speed (more precisely, the phase velocity of the wave)

Thus the speed of an electromagnetic wave is:

$$v = \frac{1}{\sqrt{\epsilon\mu}}$$

In vacuum $\mu_r = \epsilon_r = 1$, so the speed of light is:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.997924580 \times 10^8 \text{ m/s}$$

For most optical materials, $\mu_r = 1$, so:

$$v = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{n}$$

Where $n = \sqrt{\epsilon_r}$ is the familiar index of refraction.

Plane Wave Solutions:

The simplest solution to Maxwell's wave equation is, of course, the simple harmonic plane wave

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\omega t - \vec{k} \cdot \vec{r}) \quad \vec{E}_0 = \text{constant}$$

Where

\vec{k} = propagation vector or wave vector

$$|\vec{k}| = \frac{2\pi}{\lambda}, \lambda = \text{wavelength}$$

It should be noted that, strictly speaking, the plane wave is an unphysical solution—it has infinite extent in both space and time. Indeed, it is well to remember that just because a particular mathematical solution exists does not mean it can exist in physical reality. So why is the plane solution so useful? Two reasons:

1. There are physical situations which are very approximated by this solution. (e.g. the central region of a well-collimated laser beam)
2. These solutions can serve us as a mathematical basis set for expanding realistic waves in. (We'll come back to this later).

The electric field is a measurable quantity, and hence must be represented as a real number. It is convenient, however, to write $\vec{E}(\vec{r}, t) = \vec{E}_0 \exp[i(\omega t - \vec{k} \cdot \vec{r})]$ and then take the real part as the physically relevant field.

Plugging into the wave equation, we have:

$$\nabla^2 \vec{E} = \vec{E}_0 \nabla^2 \exp[i(\omega t - \vec{k} \cdot \vec{r})] = \vec{E}_0 \nabla \cdot \{-ik \exp[i(\omega t - \vec{k} \cdot \vec{r})]\} = -k^2 \vec{E}$$

And:

$$\epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \epsilon\mu \vec{E}$$

So

$$k^2 = \omega^2 \epsilon\mu$$

This is called a “dispersion relation”, which relates the wave vector to the frequency.

Applying Maxwell's equations to the plane wave, we have:

$$\nabla \cdot \vec{E} = 0$$

=>

$$\nabla \cdot \{\vec{E}_0 \exp[i(\omega t - \vec{k} \cdot \vec{r})]\} = \vec{E}_0 \cdot \nabla \exp[i(\omega t - \vec{k} \cdot \vec{r})] = -ik \vec{E}_0 \exp[i(\omega t - \vec{k} \cdot \vec{r})] = 0$$

$$\text{Or: } \vec{k} \cdot \vec{E} = 0$$

=>The field is transverse to the direction of propagation (same is true for the B field).

From the curl equation:

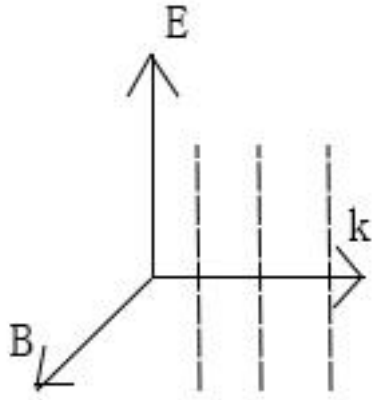
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

=>

$$-ik \times \vec{E} = -i\omega \vec{B}$$

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

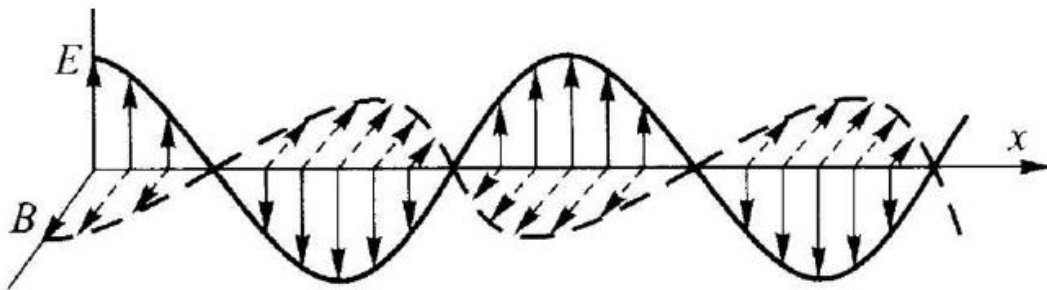
Thus the plane wave has the following structure:



Here shows the wave fronts (dotted line)

(Note that \vec{k} , \vec{E} , and \vec{B} form a right-handed orthogonal set)

Also, E and B are in phase:



$$\text{Magnitudes: } |\vec{B}| = \frac{|\vec{k}|}{\omega} |\vec{E}| = \frac{\sqrt{\omega^2 \epsilon \mu}}{\omega} |\vec{E}| = \frac{n}{c} |\vec{E}|$$

A summary of Electromagnetic Theory:

Electromagnetic theory is a discipline concerned with the study of charges at rest and in motion. Electromagnetic principles are fundamental to the study of electrical engineering and physics. Electromagnetic theory is also indispensable to the understanding, analysis and design of various electrical, electromechanical and electronic systems.

Electromagnetic Wave Equation

Recall that in a "simple" dielectric material, we derived the wave equations:

$$\nabla^2 \vec{E} - \mu\epsilon \ddot{\vec{E}} = 0 \quad (1)$$

$$\nabla^2 \vec{B} - \mu\epsilon \ddot{\vec{B}} = 0 \quad (2)$$

To derive these equations, we used Maxwell's equations with the assumptions that the charge density ρ and current density \vec{J} were zero, and that the permeability μ and permittivity ϵ were constants.

We found that the above equations had plane-wave solutions, with phase velocity:

$$v = \frac{1}{\sqrt{\mu\epsilon}} \quad (3)$$

Maxwell's equations imposed additional constraints on the directions and relative amplitudes of the electric and magnetic fields.

1

Electromagnetic Wave Equation in Conductors

How are the wave equations (and their solutions) modified for the case of electrically conducting media?

We shall restrict our analysis to the case of ohmic conductors, which are defined by:

$$\vec{J} = \sigma \vec{E} \quad (4)$$

where σ is a constant, the conductivity of the material.

All we need to do is substitute from equation (4) into Maxwell's equations, then proceed as for the case of a dielectric...

Plane Monochromatic Wave in a Conducting Material

The wave equation for the electric field in a conducting material is (11):

$$\nabla^2 \vec{E} - \mu\sigma \dot{\vec{E}} - \mu\epsilon \ddot{\vec{E}} = 0 \quad (12)$$

Let us try a solution of the same form as before:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k} \cdot \vec{r})} \quad (13)$$

Remember that to find the physical field, we have to take the real part. Substituting (13) into the wave equation (11) gives the dispersion relation:

$$-k^2 - j\omega\mu\sigma + \omega^2\mu\epsilon = 0 \quad (14)$$

Compared to the dispersion relation for a dielectric, the new feature is the presence of an imaginary term in σ . This means the relationship between the wave vector \vec{k} and the frequency ω is a little more complicated than before.

4

Plane Monochromatic Wave in a Conducting Material

In our "simple" conductor, Maxwell's equations take the form:

$$\nabla \cdot \vec{E} = 0 \quad (5)$$

$$\nabla \cdot \vec{B} = 0 \quad (6)$$

$$\nabla \times \vec{E} = -\dot{\vec{B}} \quad (7)$$

$$\nabla \times \vec{B} = \mu\epsilon \dot{\vec{E}} + \mu\vec{J} \quad (8)$$

where \vec{J} is the current density. Assuming an ohmic conductor, we can write:

$$\vec{J} = \sigma \vec{E} \quad (9)$$

so equation (8) becomes:

$$\nabla \times \vec{B} = \mu\epsilon \dot{\vec{E}} + \mu\sigma \vec{E} \quad (10)$$

Taking the curl of equation (7) and making appropriate substitutions as before, we arrive at the wave equation:

$$\nabla^2 \vec{E} - \mu\sigma \dot{\vec{E}} - \mu\epsilon \ddot{\vec{E}} = 0 \quad (11)$$

Plane Monochromatic Wave in a Conducting Material

From the dispersion relation (14), we can expect the wave vector \vec{k} to have real and imaginary parts. Let us write:

$$\vec{k} = \vec{\alpha} - j\vec{\beta} \quad (15)$$

for parallel real vectors $\vec{\alpha}$ and $\vec{\beta}$.

Substituting (15) into the dispersion relation (14) and taking real and imaginary parts, we find:

$$\alpha = \omega\sqrt{\mu\epsilon} \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} \right]^{1/2} \quad (16)$$

and:

$$\beta = \frac{\omega\mu\sigma}{2\alpha} \quad (17)$$

Equations (16) and (17) give the real and imaginary parts of the wave vector \vec{k} in terms of the frequency ω , and the material properties μ , ϵ and σ .

5

Plane Monochromatic Wave in a Conducting Material

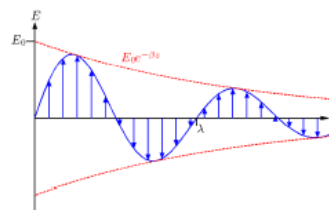
Using equation (15) the solution (13) to the wave equation in a conducting material can be written:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha} \cdot \vec{r})} e^{-\vec{\beta} \cdot \vec{r}} \quad (18)$$

The first exponential factor, $e^{j(\omega t - \vec{\alpha} \cdot \vec{r})}$ gives the usual plane-wave variation of the field with position \vec{r} and time t ; note that the conductivity of the material affects the wavelength for a given frequency.

The second exponential factor, $e^{-\vec{\beta} \cdot \vec{r}}$ gives an exponential decay in the amplitude of the wave...

Plane Monochromatic Wave in a Conducting Material



Plane Monochromatic Wave in a Conducting Material

In a "simple" non-conducting material there is no exponential decay of the amplitude: electromagnetic waves can travel for ever, without any loss of energy.

If the wave enters an electrical conductor, however, we can expect very different behaviour. The electrical field in the wave will cause currents to flow in the conductor. When a current flows in a conductor (assuming it is not a superconductor) there will be some energy changed into heat. This energy must come from the wave. Therefore, we expect the wave gradually to decay.

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Plane Monochromatic Wave in a Conducting Material

The varying electric field must have a magnetic field associated with it. Presumably, the magnetic field has the same wave vector and frequency as the electric field: this is the only way we can satisfy Maxwell's equations for all positions and times. Therefore, we try a solution of the form:

$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{j(\omega t - \vec{k} \cdot \vec{r})} \quad (19)$$

Now we use Maxwell's equation (7):

$$\nabla \times \vec{E} = -\dot{\vec{B}} \quad (20)$$

which gives:

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \quad (21)$$

or:

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \quad (22)$$

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Plane Monochromatic Wave in a Conducting Material

The magnetic field in a wave in a conducting material is related to the electric field by (22):

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \quad (23)$$

As in a non-conducting material, the electric and magnetic fields are perpendicular to the direction of motion (the wave is a transverse wave) and are perpendicular to each other.

But there is a new feature, because the wave vector is complex.

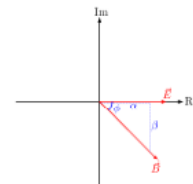
In a non-conducting material, the electric and magnetic fields were in phase: the expressions for the fields both had the same phase angle ϕ_0 . In complex notation, the complex phase angles of the field amplitudes \vec{E}_0 and \vec{B}_0 were the same.

In a conductor, the complex phase of \vec{k} gives a phase difference between the electric and magnetic fields

Plane Monochromatic Wave in a Conducting Material

In a conducting material, there is a difference between the phase angles of \vec{E}_0 and \vec{B}_0 , given by the phase angle ϕ of \vec{k} . This is:

$$\tan \phi = \frac{\beta}{\alpha} \quad (24)$$



Plane Monochromatic Wave in a Poor Conductor

Let us consider the special case of a good insulator. In this case:

$$\sigma \ll \omega\epsilon \quad (25)$$

From equation (16), we then have:

$$\alpha \approx \omega\sqrt{\mu\epsilon} \quad (26)$$

and from equation (17) we have:

$$\beta \approx \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}} = \frac{\alpha}{2} \frac{\sigma}{\omega\epsilon} \quad (27)$$

It follows that $\beta \ll \alpha$. We recover the same situation as in the case of a non-conducting material. The decay of the wave is very slow (in terms of the number of wavelengths); the magnetic and electric components of the wave are approximately in phase ($\phi \approx 0$), and are related by:

$$B_0 \approx \frac{\alpha}{\omega} E_0 \approx \frac{E_0}{v_p} \quad (28)$$

where the phase velocity v_p is, as before, given by $v_p = 1/\sqrt{\mu\epsilon}$.

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Plane Monochromatic Wave in a Good Conductor

Let us consider the special case of a very good conductor. In this case:

$$\sigma \gg \omega\epsilon \quad (29)$$

From equation (16), we then have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} \quad (30)$$

and from equation (17) we have:

$$\beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} \approx \alpha \quad (31)$$

In the case of a very good conductor, the real and imaginary parts of the wave vector \vec{k} become equal. This means that the decay of the wave is very fast in terms of the number of wavelengths.

Note that the vectors $\vec{\alpha}$ and $\vec{\beta}$ have the same units as \vec{k} , i.e. meters^{-1} .

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Phase Velocity in a Good Conductor

The electric field in the wave varies as (18):

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha} \cdot \vec{r})} e^{-\vec{\beta} \cdot \vec{r}} \quad (32)$$

The phase velocity is the velocity of a point that stays in phase with the wave. Consider a wave moving in the $+z$ direction:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \alpha z)} e^{-\beta z} \quad (33)$$

For a point staying at a fixed phase, we must have:

$$\omega t - \alpha z(t) = \text{constant} \quad (34)$$

So the phase velocity is given by:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\alpha} \quad (35)$$

But note that in a good conductor, α is itself a function of ω ...

Phase Velocity in a Good Conductor

For a poor conductor ($\sigma \ll \omega\epsilon$), we have:

$$\alpha \approx \omega\sqrt{\mu\epsilon} \quad (36)$$

so the phase velocity in a poor conductor is:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\epsilon}} \quad (37)$$

If μ and ϵ are constants (i.e. are independent of ω) then the phase velocity is independent of the frequency: there is no dispersion.

However, in a good conductor ($\sigma \gg \omega\epsilon$), we have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\mu\epsilon} \sqrt{\frac{\omega\sigma}{2\epsilon}} \quad (38)$$

Then the phase velocity is given by:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\frac{2\omega\epsilon}{\sigma}} \quad (39)$$

The phase velocity depends on the frequency: there is dispersion!

Phase Velocity and Group Velocity

The presence of dispersion means that the group velocity v_g (the velocity of a wave pulse) can differ from the phase velocity v_p (the velocity of a point staying at a fixed phase of the wave).

To understand what this means, consider the superposition of two waves with equal amplitudes, both moving in the $+z$ direction, and with similar wave numbers:

$$E_x = E_0 \cos(\omega_+ t - [k_0 + \Delta k] z) + E_0 \cos(\omega_- t - [k_0 - \Delta k] z) \quad (40)$$

Using a trigonometric identity:

$$\cos A + \cos B \equiv 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \quad (41)$$

the electric field can be written:

$$E_x = 2E_0 \cos(\omega_0 t - k_0 z) \cos(\Delta\omega t - \Delta k z) \quad (42)$$

where:

$$\omega_0 = \frac{1}{2}(\omega_+ + \omega_-) \quad \Delta\omega = \omega_+ - \omega_- \quad (43)$$

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Phase Velocity and Group Velocity

We have written the total electric field in our superposed waves as (42):

$$E_x = 2E_0 \cos(\omega_0 t - k_0 z) \cos(\Delta\omega t - \Delta k z) \quad (44)$$

Assuming that $\Delta k \ll k_0$, the first trigonometric factor represents a wave of (short) wavelength $2\pi/k_0$ and phase velocity:

$$v_p = \frac{\omega_0}{k_0} \quad (45)$$

while the second trigonometric factor represents a modulation of (long) wavelength $2\pi/\Delta k$, which travels with velocity:

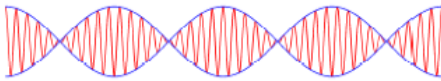
$$v_g = \frac{\Delta\omega}{\Delta k} \quad (46)$$

v_g is called the group velocity. Since $\Delta\omega$ represents the change in frequency that corresponds to a change Δk in wave number, we can write:

$$v_g = \frac{d\omega}{dk} \quad (47)$$

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Group Velocity and Energy Flow



The red wave moves with the phase velocity v_p ; the modulation (represented by the blue line) moves with group velocity v_g .

Since the energy in a wave depends on the local amplitude of the wave, the energy in the wave is carried at the group velocity v_g .

Phase Velocity and Group Velocity

If there is no dispersion, then the phase velocity is independent of frequency:

$$v_p = \frac{\omega}{k} = \text{constant} \quad (48)$$

and the group velocity is equal to the phase velocity:

$$v_g = \frac{d\omega}{dk} = v_p \quad (49)$$

In the absence of dispersion, a modulation resulting from the superposition of two waves with similar frequencies will travel at the same speed as the waves themselves.

However, if there is dispersion, then the group velocity can differ from the phase velocity...

Group Velocity of an EM Wave in a Good Conductor

The dispersion relation for an electromagnetic wave in a good conductor is, from (38):

$$\omega = \frac{1}{\mu\epsilon} \frac{2\epsilon}{\sigma} \alpha^2 \quad (50)$$

where α is the real part of the wave vector. The group velocity is then:

$$\begin{aligned} v_g &= \frac{d\omega}{d\alpha} \\ &\approx \frac{1}{\mu\epsilon} \frac{4\epsilon}{\sigma} \alpha \\ &\approx \frac{2}{\sqrt{\mu\epsilon}} \sqrt{\frac{2\omega\epsilon}{\sigma}} \end{aligned} \quad (51)$$

Comparing with equation (39) for the phase velocity of an electromagnetic wave in a good conductor, we find that:

$$v_g \approx 2v_p \quad (52)$$

In other words, the group velocity is approximately twice the phase velocity.

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The Skin Depth of a Good Conductor

The real part, α , of the wave vector k in a conductor gives the wavelength of the wave. β measures the distance that the wave travels before its amplitude falls to $1/e$ of its original value. Let us write the solution (18) for a wave travelling in the z direction in a good conductor as:

$$\vec{E}(\vec{r}, t) = \vec{E}_0(\vec{r}) e^{j(\omega t - \vec{\alpha} \cdot \vec{r})} \quad (53)$$

where:

$$\vec{E}_0(\vec{r}) = \vec{E}_0 e^{-\beta \cdot \vec{r}} \quad (54)$$

The amplitude of the wave falls by a factor $1/e$ in a distance $1/\beta$. We define the *skin depth* δ :

$$\delta = \frac{1}{\beta} \quad (55)$$

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The Skin Depth of a Good Conductor

From equation (31), we see that for a good conductor ($\sigma \gg \omega\epsilon$), the skin depth is given by:

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \quad (56)$$

For example, consider silver, which has conductivity $\sigma \approx 6.30 \times 10^7 \Omega^{-1} \text{m}^{-1}$, and permittivity $\epsilon \approx \epsilon_0 \approx 8.85 \times 10^{-12} \text{Fm}^{-1}$.

For radiation of frequency 10^{10} Hz, the "good conductor" condition is satisfied, and the skin depth of the radiation is approximately 0.6 micron (0.6×10^{-6} m).

Note that in vacuum, the wavelength of radiation of frequency 10^{10} Hz is about 3 cm; but in silver, the wavelength is:

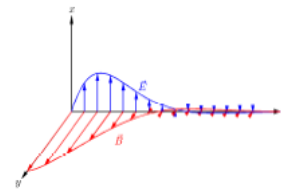
$$\lambda = \frac{2\pi}{\alpha} \approx 2\pi\delta \approx 4 \text{ micron} \quad (57)$$

Plane Monochromatic Wave in a Good Conductor

The phase difference between the electric and magnetic fields in a good conductor is given by:

$$\tan \phi = \frac{\beta}{\alpha} \approx 1 \quad (58)$$

So the phase difference is approximately 45° .



<p style="text-align: center;">EM Wave Impedance in a Good Conductor</p> <hr/> <p>Using the plane wave solutions:</p> $\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k} \cdot \vec{r})} \quad (59)$ $\vec{B}(\vec{r}, t) = \vec{B}_0 e^{j(\omega t - \vec{k} \cdot \vec{r})} \quad (60)$ <p>in Maxwell's equation:</p> $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (61)$ <p>and using also the relation $\vec{B} = \mu \vec{H}$, we find the relation between the electric field and magnetic intensity:</p> $\vec{k} \times \vec{E}_0 = \omega \mu \vec{H}_0 \quad (62)$ <p>The vectors \vec{k}, \vec{E}_0 and \vec{H}_0 are mutually perpendicular. Therefore, we can write for the wave impedance:</p> $Z = \frac{E_0}{H_0} = \frac{\omega \mu}{\alpha - j\beta} \quad (63)$ <hr/> <p style="text-align: center;">24</p>	<p style="text-align: center;">EM Wave Impedance in a Good Conductor</p> <hr/> <p>In a good conductor ($\sigma \gg \omega \epsilon$), we have (31):</p> $\alpha \approx \beta \approx \sqrt{\frac{\omega \mu \sigma}{2}} \quad (64)$ <p>It then follows that the wave impedance (63) in a good conductor is given by:</p> $Z \approx \frac{1}{1-j} \sqrt{\frac{2\omega \mu}{\sigma}} = (1+j) \sqrt{\frac{\omega \mu}{2\sigma}} \quad (65)$ <p>Note that the impedance is now a complex number. As we shall see later, the behaviour of waves on a boundary depends on the impedances of the media on either side of the boundary.</p> <p>The complex phase of the impedance will tell us about the phases of the waves reflected from and transmitted across the boundary.</p> <hr/> <p style="text-align: center;">25</p>
<p style="text-align: center;">Energy Densities in an EM Wave in a Good Conductor</p> <hr/> <p>The time averaged energy densities in the electric and magnetic fields are:</p> $\langle U_E \rangle_t = \frac{1}{2} \epsilon \langle \vec{E}^2 \rangle_t = \frac{1}{4} \epsilon E_0^2 e^{-2\beta \cdot \vec{r}} \quad (66)$ $\langle U_H \rangle_t = \frac{1}{2} \mu \langle \vec{H}^2 \rangle_t = \frac{1}{4} \mu H_0^2 e^{-2\beta \cdot \vec{r}} \quad (67)$ <p>The ratio is:</p> $\frac{\langle U_E \rangle_t}{\langle U_H \rangle_t} = \frac{\epsilon E_0^2}{\mu H_0^2} = \frac{\epsilon}{\mu} Z ^2 \quad (68)$ <p>In a good conductor, the square of the magnitude of the impedance is:</p> $ Z ^2 \approx \frac{\omega \mu}{\sigma} \quad (69)$ <p>Hence, in a good conductor, most of the energy is in the magnetic field:</p> $\frac{\langle U_E \rangle_t}{\langle U_H \rangle_t} \approx \frac{\omega \epsilon}{\sigma} \ll 1 \quad (70)$ <hr/>	<p style="text-align: center;">Complex Conductivity: the Drude Model</p> <hr/> <p>So far, we have assumed that the conductivity is a real number, and is independent of frequency. This is approximately true for low frequencies.</p> <p>However, at high frequencies (visible frequencies and above) the behaviour of electromagnetic waves in many conductors is best described by a complex conductivity that is a function of frequency. Recall that the conductivity gives the relationship between the current density and the electric field:</p> $\vec{J} = \sigma \vec{E} \quad (71)$ <p>So a complex conductivity indicates a phase difference between the current density and an oscillating electric field.</p> <p>A model to describe this behaviour, based on the dynamics of the free electrons in the conductor, was developed in the 1900's by the German physicist Paul Drude. The detailed behaviour can get quite complicated, so we will just sketch out the main ideas.</p> <hr/>

Complex Conductivity: the Drude Model

Electrical conductors have both bound and free electrons. The bound electrons behave the same way as in a dielectric, and are subject to a binding force $-Kx$. The free electrons have no binding force. The equation of motion for free electrons in an electromagnetic wave is therefore:

$$\ddot{x} + \Gamma \dot{x} = \frac{e}{m} E_0 e^{j\omega t} \quad (72)$$

which has the solution:

$$x = \frac{e/m}{-\omega^2 + j\omega\Gamma} E_0 e^{j\omega t} \quad (73)$$

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Complex Conductivity: the Drude Model

Now, the current density J depends on the conductivity σ :

$$J = \sigma E = Ne\dot{x} \quad (74)$$

where N is the number of free electrons per unit volume. From equation (73), we find:

$$\dot{x} = \frac{j\omega/m}{-\omega^2 + j\omega\Gamma} E_0 e^{j\omega t} \quad (75)$$

Therefore, we can write for the conductivity:

$$\sigma = \frac{j\omega Ne^2/m}{-\omega^2 + j\omega\Gamma} = \frac{Ne^2/m}{\Gamma + j\omega} \quad (76)$$

The conductivity is a complex number:

$$\sigma = \sigma_1 - j\sigma_2 \quad (77)$$

where:

$$\sigma_1 = \frac{Ne^2\Gamma/m}{\Gamma^2 + \omega^2}, \quad \sigma_2 = \frac{Ne^2\omega/m}{\Gamma^2 + \omega^2} \quad (78)$$

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Complex Conductivity: the Drude Model

Note that we can relate the "damping constant" Γ of the electron motion to the dc conductivity σ_0 (the conductivity at zero frequency):

$$\sigma_0 = \lim_{\omega \rightarrow 0} \sigma = \frac{Ne^2}{\Gamma m} \quad (79)$$

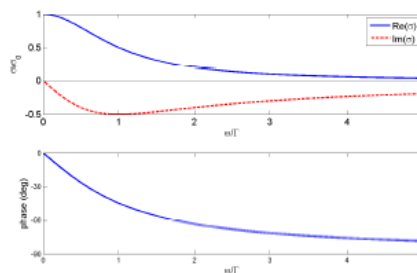
In terms of σ_0 , the conductivity can be written:

$$\sigma = \frac{\sigma_0}{1 + j\omega/\Gamma} \quad (80)$$

Equation (80) describes how the conductivity of a conductor varies with frequency, and is the main result of the Drude model. The constant σ_0 can be determined by experiment; if N , e and m are known, then Γ can then be calculated from (79):

$$\Gamma = \frac{Ne^2}{\sigma_0 m} \quad (81)$$

Complex Conductivity: the Drude Model



Complex Conductivity: the Drude Model

- At very low frequencies:

$$\omega \rightarrow 0, \quad \sigma_1 \rightarrow \frac{Ne^2}{m\Gamma}, \quad \sigma_2 \rightarrow 0 \quad (82)$$

i.e. σ is real and constant, as for dc conductivity.

- At low frequencies ($\omega \ll \sigma/\epsilon$, up to the infra-red range) the free electron term dominates.
- In the visible region ($\omega \approx \sigma/\epsilon$), both terms contribute, and the formulae (78) for the conductivity agree quite well with the experimental results.
- At high frequencies ($\omega \gg \sigma/\epsilon$, X-rays and γ -rays) the free electron term is small, and the material behaves like a dielectric.

Summary of Part 3

You should be able to:

- Derive, from Maxwell's equations, the wave equations for the electric and magnetic fields in conducting media.
- Explain the origin of the "good conductor condition" $\sigma \gg \omega\epsilon$ for an electromagnetic plane wave.
- Derive the relationships (amplitude, phase, direction) between the electric and magnetic fields in a plane wave in conducting media.
- Derive expressions for the phase and group velocities of an electromagnetic wave in a good conductor.
- Show that in a conductor the amplitude decays exponentially and explain what happens to the energy of the wave.
- Derive an expression for the "skin depth" in the case of a plane wave travelling through a conductor.
- Explain that when an electromagnetic wave moves through a conducting medium, the conductivity of the medium can be written as a complex number, with a dependence on the frequency of the wave.